The Null Space Property of the Weighted $\ell_r - \ell_1$ Minimization

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Abstract. The null space property (NSP), which relies merely on the null space of the sensing matrix column space, has drawn numerous interests in sparse signal recovery. This article studies NSP of the weighted $\ell_r - \ell_1$ minimization. Several versions of NSP of the weighted $\ell_r - \ell_1$ minimization including the weighted $\ell_r - \ell_1$ NSP, the weighted $\ell_r - \ell_1$ stable NSP, the weighted $\ell_r - \ell_1$ robust NSP, and the $\ell_q$ weighted $\ell_r - \ell_1$ NSP for $1 \leq q \leq 2$, are proposed, as well as the associating considerable results are derived. Under these NSP, sufficient conditions for the recovery of (sparse) signals with the weighted $\ell_r - \ell_1$ minimization are established. Furthermore, we show that to some extent, the weighted $\ell_r - \ell_1$ stable NSP is weaker than the restricted isometric property (RIP). And the RIP condition we obtained is better than that of Zhou Z. (2022).

Key words. Compressed sensing; null space property; the weighted $\ell_r - \ell_1$ minimization; sparse signal recovery.

1 Introduction

Recovery of an unknown (nearly) sparse signal $x \in \mathbb{R}^n$ from the linear measurements corrupted by noise

$$b = \Phi x + e,$$

where $b, e \in \mathbb{R}^m$, $\Phi \in \mathbb{R}^{m \times n} (m \ll n)$, has been attracting broad and hot attention, see [1, 16].

Recently, the weighted $\ell_r - \ell_1$ minimization approach is proposed in [2] to reconstruct $x$. This method promotes sparsity, better the recovery performance, and has virtues for sparse signal reconstruction through experiments compared to the state-of-art algorithms. The constrained weighted $\ell_r - \ell_1$ minimization approach is defined by [2]

$$\min_{\hat{x} \in \mathbb{R}^n} \|\hat{x}\|_r^r - \alpha \|\hat{x}\|_1^r \text{ subject to } \|\Phi \hat{x} - b\|_2 \leq \epsilon,$$

(1.2)
where \( \|x\|_r = (\sum_{i=1}^{n} |x_i|^r)^{1/r} \), \( r \in (0, 1] \), \( \alpha \in [0, 1] \), and \( \epsilon \in (0, \infty) \). All over the paper, suppose that \( \alpha \neq 1 \) in the case of \( r = 1 \). It is obvious that (1.2) returns to the usual \( \ell_1 \) minimization approach in the case of \( \alpha = 0 \), which permits accurate reconstruction from fewer number of samples than that using \( \ell_1 \) minimization method, for more details, see [3–8, 18].

In [2, 10, 11, 13], based on the \( r \)-restricted isometry property (RIP) [8], researchers have performed theoretical and experimental analysis of the reconfigurability properties of (1.2).

**Definition 1.1.** ([8]) For integer \( k \in (0, \infty) \) and \( r \in (0, 1] \), we define the \( k \)-th \( r \)-restricted isometry constant (RIC) \( \delta_k \) of a matrix \( \Phi \in \mathbb{R}^{m \times n} \) as the smallest \( \delta \in [0, \infty) \) satisfying the following inequality
\[
(1 - \delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta)\|x\|_2^2 \tag{1.3}
\]
for all \( k \)-sparse vectors \( x \in \mathbb{R}^n \) (i.e., the number of nonzero elements of \( x \) is smaller than \( k \)).

The theoretical reconstruction results for the weighted \( \ell_r - \ell_1 \) minimization approach (1.2) based on RIP and \( q \)-ratio constrained minimal singular values [9] are presented in [2]. Recently, Cai [10] extended the model (1.2) to the recovery of block structure signal and low-rank matrix. For more related work, we refer readers to see [11] for involving prior support information, [12] for the mutual coherence based sufficient condition, [13] for the based-RIP and higher-order RIP analysis results. Another extensively utilizing framework in compressive sensing is the null space property (NSP). For instance, the NSP of \( \ell_1 \) minimization and \( \ell_r \) minimization was introduced by [14–17] and employing these properties, authors provided the characterizations of sufficient conditions for the accurate reconstruction of sparse signal with \( \ell_1 \) minimization method and \( \ell_r \) minimization method. In addition, [19] and [18] separately exploited NSP of \( \ell_1 \) minimization and \( \ell_r \) minimization to give the proof of sufficient conditions based on RIP for the accurate reconstruction of sparse vectors using \( \ell_1 \) minimization method and \( \ell_r \) minimization method. For more related studies concerning NSP, see e.g. [20, 21].

In this paper, we discuss the different versions of NSP regarding the weighted \( \ell_r - \ell_1 \) minimization approach (1.2). Sufficient conditions for stability and robustness of the solutions to the model (1.2) are established under these NSP conditions. Our purpose here is only to give a theoretical analysis regarding the model (1.2).

The remainder of this article is constructed as follows. In Section 2, we firstly provide the weighted \( \ell_r - \ell_1 \) NSP, and based on it, we depict the exact recovery of \( k \)-sparse signal with (1.2). Afterward, we present the weighted \( \ell_r - \ell_1 \) stable NSP and establish its equivalent modus, then we show that it is not stronger than the RIP to a certain degree. In Section 3, we separately consider the weighted \( \ell_r - \ell_1 \) robust NSP and the \( \ell_q \) weighted \( \ell_r - \ell_1 \) robust NSP. We give the efficient description of stable and robust recovery of signal via the weighted \( \ell_r - \ell_1 \) minimization (1.2) under these null space properties. The important lemmas that will be used in the proofs of main results are deferred in the appendix.

**Notations:** Set \( [n] = \{1, 2, \cdots, n\} \). For the index set \( T \subseteq [n] \), \( x_T \) stands for the vector that is equal to \( x \) on \( T \) and 0 on \( T^c = [n]\setminus T \) as well as \( x_T[i] \) indicates the \( i \)-th element of \( x_T \), where \( i \in [n] \). Denote by \( |T| \) the number of entries of the set \( T \). \( \text{supp}(x) = \{i : x_i \neq 0\} \) represents the support of \( x \).
2 Weighted $\ell_r - \ell_1$ Stable Null Space Property

In this section, we begin with proposing NSP of order $k$ of the weighted $\ell_r - \ell_1$ minimization to resolve the reconstruction of sparse signals using (1.2).

Definition 2.1. For any set $T \subseteq [n]$ with $|T| \leq k$, the matrix $\Phi$ is said to fulfill the weighted $\ell_r - \ell_1$ null space property of order $k$ if

$$\|w_T\|_r + \alpha\|w\|_1 < \|w_T\|_r$$

(2.4)

for all $w \in \mathcal{N}(\Phi) \setminus \{0\}$ ($\mathcal{N}(\Phi)$ is the null space of $\Phi$).

Definition 2.1 reduces to the $\ell_r$ null space property [17] [16] when $\alpha = 0$.

We now show that the connection between the weighted $\ell_r - \ell_1$ null space property and accurate reconstruction of sparse signals by (1.2).

Theorem 2.2. Let $x$ be a known signal with $|\text{supp}(x)| \leq k$. Suppose that the matrix $\Phi$ fulfills the weighted $\ell_r - \ell_1$ null space property, then $x$ is the unique solution to (1.2) with $b = \Phi x$ and $\epsilon = 0$.

Proof. Assume that the matrix $\Phi$ satisfies the weighted $\ell_r - \ell_1$ null space property. We suppose that $x$ is a $k$-sparse signal. Set $T = \text{supp}(x)$, then $|T| \leq k$. Set further $u \neq x$ be satisfying $\Phi u = \Phi x$. Then $w = u - x \in \mathcal{N}(\Phi) \setminus \{0\}$ and

$$\|x\|_r - \alpha\|w\|_1 \leq \|(x-u)_T\|_r + \|u_T\|_r - \alpha\|u\|_1 + \alpha\|x-u\|_1$$

$$= \|w_T\|_r + \alpha\|w\|_1 + \|u_T\|_r - \alpha\|u\|_1$$

$$< \|w_T\|_r + \|u_T\|_r - \alpha\|u\|_1$$

$$= \|u\|_r - \alpha\|u\|_1,$$

where (a) is from the inequality (2.4). The desired result is established. $\square$

In more practical context, the signals to be recovered are close to sparse signals but no totally sparse ones. In this situation, we introduce the concept of the weighted $\ell_r - \ell_1$ stable null space property to study the recovery of signals.

Definition 2.3. For any set $T \subseteq [n]$ with $|T| \leq k$, the matrix $\Phi$ is said to fulfill the weighted $\ell_r - \ell_1$ stable null space property of order $k$ with constant $\mu \in (0, 1)$ if

$$\|w_T\|_r + \alpha\|w\|_1 \leq \mu\|w_T\|_r$$

(2.5)

for all $w \in \mathcal{N}(\Phi)$.

Remark 2.4. For $w \in \mathcal{N}(\Phi)$ with $w \neq 0$, suppose that the weighted $\ell_r - \ell_1$ stable NSP of order $k$ with constant $\mu$ holds, then $\|w_T\|_r + \alpha\|w\|_1 \leq \mu\|w_T\|_r < \|w_T\|_r$, which deduces that the weighted $\ell_r - \ell_1$ NSP of order $k$ holds, the converse doesn’t hold. Namely, Definition 2.1 is weaker than Definition 2.3 when $w \in \mathcal{N}(\Phi) \setminus \{0\}$. 

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Definition 2.3 provides the key assumption for proving our main result. Before that, we deduce a proposition which will be used in the proof of the main theorem.

**Proposition 2.5.** Set $T \subseteq [n]$ with $|T| \leq k$. Assume that the matrix $\Phi$ fulfills the weighted $\ell_r - \ell_1$ stable NSP of order $k$ with constant $\mu \in (0, 1)$. Then, for any vector $u, v \in \mathbb{R}^n$ satisfying $\Phi u = \Phi v$,

$$
\|v - u\|_r + \alpha \|v - u\|_1^r \leq \frac{1 + \mu}{1 - \mu} \left( \|v\|_r^r - \alpha \|v\|_1^r - (\|u\|_r^r - \alpha \|u\|_1^r) + 2\|w_T\|_r^r \right),
$$

(2.6)

holds.

Conversely, in case (2.6) holds, we have $\Phi$ fulfills the below inequality

$$
\|w_T\|_r^r + \frac{1}{2} \alpha [(1 - \mu)\|w\|_1^r + (1 + \mu)\|w_T\|_r^r - (1 + \mu)\|w_T\|_1^r] \leq \mu\|w_T\|_r^r.
$$

(2.7)

**Proof.** Assume that $\Phi$ fulfills the weighted $\ell_r - \ell_1$ stable NSP of order $k$ with constant $\mu \in (0, 1)$. For $u, v \in \mathbb{R}^n$ with $\Phi u = \Phi v$, note that $w = v - u \in \mathcal{N}(\Phi)$, then the weighted $\ell_r - \ell_1$ stable NSP implies

$$
\|w_T\|_r^r + \alpha \|w\|_1^r \leq \mu\|w_T\|_r^r.
$$

(2.8)

By exploiting Lemma 3.7 and (2.8), we have

$$
\|w_T\|_r^r \leq \|w\|_r^r - \alpha \|v\|_1^r - (\|u\|_r^r - \alpha \|u\|_1^r) + \|w_T\|_r^r + \alpha \|w\|_1^r + 2\|w_T\|_r^r
$$

$$
\leq \|w\|_r^r - \alpha \|v\|_1^r - (\|u\|_r^r - \alpha \|u\|_1^r) + \mu\|w_T\|_r^r + 2\|w_T\|_r^r.
$$

(2.9)

Due to $\mu \in (0, 1)$, we get

$$
\|w_T\|_r^r \leq \frac{1}{1 - \mu} \left( \|w\|_r^r - \alpha \|v\|_1^r - (\|u\|_r^r - \alpha \|u\|_1^r) + 2\|w_T\|_r^r \right).
$$

(2.10)

By (2.10) and again utilizing (2.8), we have

$$
\|w\|_r^r + \alpha \|w\|_1^r \leq \|w_T\|_r^r + \alpha \|w\|_1^r + \|w_T\|_r^r \leq (1 + \mu)\|w_T\|_r^r
$$

$$
\leq \frac{1 + \mu}{1 - \mu} \left( \|w\|_r^r - \alpha \|v\|_1^r - (\|u\|_r^r - \alpha \|u\|_1^r) + 2\|w_T\|_r^r \right).
$$

Conversely, suppose that the matrix $\Phi$ meets (2.6) for any vector $u, v \in \mathbb{R}^n$ with $\Phi u = \Phi v$. For a given $w \in \mathcal{N}(\Phi)$, because $\Phi w_T = \Phi(-w_T)$, put $v = w_T$ and $u = -w_T$, so $w_T = 0$. By (2.6), we get

$$
\|w\|_r^r + \alpha \|w\|_1^r = \|w_T\|_r^r + \alpha \|w_T\|_r^r + \|w_T\|_1^r \leq \frac{1 + \mu}{1 - \mu} \left( \|w_T\|_r^r - \alpha \|w_T\|_1^r - (\|w_T\|_r^r - \alpha \|w_T\|_1^r) \right).
$$

The above inequality could be rephrased as

$$
(1 - \mu)(\|w_T\|_r^r + \|w_T\|_r^r) + \alpha (1 - \mu)\|w\|_1^r \leq (1 + \mu) \left( \|w_T\|_r^r - \alpha \|w_T\|_1^r - (\|w_T\|_r^r - \alpha \|w_T\|_1^r) \right).
$$

Therefore,

$$
\|w_T\|_r^r + \alpha \frac{1}{2} [(1 - \mu)\|w\|_1^r + (1 + \mu)\|w_T\|_1^r - (1 + \mu)\|w_T\|_1^r] \leq \mu\|w_T\|_r^r.
$$

We complete the proof.
With Proposition 2.5, we are in the position to show our main theorem which presents the characterization of stable recovery of the signal using (1.2) with \( b = \Phi x \) and \( \epsilon = 0 \).

**Theorem 2.6.** Let \( x \) be the real-world signal in \( b = \Phi x \), \( \hat{x} \) be the solution to the problem (1.2). Suppose that the matrix \( \Phi \) fulfills the weighted \( \ell_r - \ell_1 \) stable NSP of order \( k \) with constant \( \mu \in (0, 1) \). We have

\[
\|x - \hat{x}\|_r \leq \left( \frac{2(1 + \mu)}{1 - \mu} \right)^{\frac{1}{2}} \sigma_k(x)_r,
\]

where

\[
\sigma_k(x)_r = \inf\{\|x - y\|_r, y \in \mathbb{R}^n \text{ and } |\text{supp}(y)| \leq k\}
\]

represents the \( \ell_r \)-error of best \( k \)-term approximation of \( x \).

**Proof.** Let \( T \) be the index set of \( k \) largest absolute coefficients of \( x \), then \( \sigma_k(x)_r = \|x_T\|_r \). Since \( \hat{x} \) is a solution of (1.2) with \( b = \Phi x \) and \( \epsilon = 0 \), then \( \Phi x = \Phi \hat{x} \) and \( \|\hat{x}\|_r = \|x\|_r \leq \|x\|_r - \alpha \|x\|_1 \leq \|x\|_r - \alpha \|x\|_1 \). Take \( v = \hat{x} \) and \( u = x \) in the inequality (2.6), then

\[
\|x - \hat{x}\|_r \leq \frac{1 + \mu}{1 - \mu} \left( \|\hat{x}\|_r - \alpha \|\hat{x}\|_1 - (\|x\|_r - \alpha \|x\|_1) + 2\|x_T\|_r^2 \right)
\]

\[
\leq \frac{2(1 + \mu)}{1 - \mu} \sigma_k(x)_r.
\]

The proof is complete. \( \square \)

On the basis of Theorem 2.6, the following corollary follows.

**Corollary 2.7.** Assume that the matrix \( \Phi \) fulfills the weighted \( \ell_r - \ell_1 \) stable NSP of order \( k \) with constant \( \mu \in (0, 1) \), then every \( k \)-sparse signal \( x \) can be accurately estimated by the weighted \( \ell_r - \ell_1 \) minimization with \( b = \Phi x \).

Corollary 2.7 gives an exact recovery guarantee for method (1.2) when the signal \( x \) is strictly \( k \)-sparse. Moreover, it is important to propose the weighted \( \ell_r - \ell_1 \) stable NSP since it is weaker than the RIP. We will observe that somehow, the RIP is able to deduce the weighted \( \ell_r - \ell_1 \) stable NSP.

**Theorem 2.8.** For any \( k \)-sparse signal \( x \), if the matrix \( \Phi \) fulfills the RIP of order \( k \) with constant \( \delta_{2k} \), where

\[
\delta_{2k} < \frac{2}{1 + \sqrt{2} + 2 \left( \frac{1 + \alpha}{1 - \alpha} \right)^{\frac{1}{2}}}
\]

then \( \Phi \) meets the weighted \( \ell_r - \ell_1 \) stable NSP of order \( k \) with constant \( \mu \), where

\[
\mu = \frac{(1 + \alpha)(1 + \sqrt{2})^r \delta_{2k}^r}{2r(1 - \delta_{2k})^r} + \alpha.
\]

**Proof.** See the appendix. \( \square \)

**Corollary 2.9.** If the matrix \( \Phi \) satisfies the restricted isometry property with constant \( \delta_{2k} \) satisfying (2.14), then every \( k \)-sparse signal \( x \) is the unique solution of (1.2) with \( b = \Phi x \).
Remark 2.10. Recently, Zhou [13] showed that an exact reconstruction can be ensured for any $k$-sparse vector by (1.2) with $\epsilon = 0$, when the RIP condition $\delta_{2k} < \tau / \sqrt{\tau^2 + \gamma}$ with $\tau = \left(\frac{k - nk^*}{k+nk^*}\right)^{1/r}$ and $\gamma = \frac{2^{1/r+\frac{1}{r}}}{k}((1 + \alpha - \alpha^2)^{-2/(r+1)} + 1)$ for $k \geq 2$ is satisfied. We compared the condition (2.14) obtained in this paper with that of Zhou [13], looking at Fig. 2.1 (a) and (b). Here, $\alpha = 2$. The red dashed line indicates that we get the upper bound, and the blue dashed line indicates the result of Zhou. From Fig. 2.1 (a), we can see that the upper bound of $\delta_{2k}$ we obtained is better than that of Zhou [13]. Observing Fig. 2.1 (b), we can see that the upper bound given in this paper is also better than that of Zhou, except that $\alpha \in [0, 0.16]$.

Fig. 2.1: Comparisons of bounds between (2.14) and Zhou [13]. (a) Upper bound of $\delta_{2k}$ versus $r$ for given $\alpha = 0.5$; (b) Upper bound of $\delta_{2k}$ versus $\alpha$ for given $r = 0.95$.

3 Weighted $\ell_r - \ell_1$ Robust Null Space Property

We will consider the model (1.2) with the linear measurements interfered by noise. In order to describe the solution of (1.2), we propose the notion of weighted $\ell_r - \ell_1$ robust null space property for the recovery formula in terms of measurement noise.

**Definition 3.1.** For any set $T \subseteq [n]$ with $|T| \leq k$, the matrix $\Phi$ is said to fulfill the weighted $\ell_r - \ell_1$ robust null space property of order $k$ with constants $\mu \in (0, 1)$ and $\rho \in (0, +\infty)$ if

$$\|w_T\|_r + \alpha\|w\|_1 \leq \mu\|w_T\|_r + \rho\|\Phi w\|_2$$

for all $w \in \mathbb{R}^n$.

Clearly, Definition 3.1 is more general than Definition 2.3 because it does not have the $w \in \mathcal{N}(\Phi)$ restriction. The weighted $\ell_r - \ell_1$ robust NSP signifies the weighted $\ell_r - \ell_1$ stable NSP in the case of $w \in \mathcal{N}(\Phi)$. From Definition 3.1 we obtain the following important result.

**Proposition 3.2.** Set $T \subseteq [n]$ with $|T| \leq k$. Assume that the matrix $\Phi$ fulfills the weighted $\ell_r - \ell_1$ robust NSP
of order $k$ with constants $\mu \in (0, 1)$ and $\rho \in (0, +\infty)$. Then, for any vector $u, v \in \mathbb{R}^n$ satisfying $\Phi u = \Phi v$,

$$\|v - u\|_r^r + \alpha \|v - u\|_1^1 \leq \frac{1 + \mu}{1 - \mu} (\|v\|_r^r - \alpha \|v\|_1^1 - (\|u\|_r^r - \alpha \|u\|_1^1) + 2\|w_{T^r}\|_r^r) + \frac{2\rho}{1 - \mu} \|\Phi w\|_2$$

(3.17)

holds.

**Proof.** Assume that $\Phi$ fulfills the weighted $\ell_r - \ell_1$ robust NSP of order $k$ with constants $\mu \in (0, 1)$ and $\rho \in (0, +\infty)$. For $u, v \in \mathbb{R}^n$ with $\Phi u = \Phi v$, putting $w = v - u$, then the weighted $\ell_r - \ell_1$ robust NSP and Lemma 3.7 deduces

$$\|w_{T^r}\|_r^r + \alpha \|w\|_1^1 \leq \mu \|w_{T^r}\|_r^r + \rho \|\Phi w\|_2. \quad (3.18)$$

$$\|w_{T^r}\|_r^r \leq \|v\|_r^r - \alpha \|v\|_1^1 - (\|u\|_r^r - \alpha \|u\|_1^1) + \|w_{T^r}\|_r^r + \alpha \|w\|_1^1 + 2\|w_{T^r}\|_r^r. \quad (3.19)$$

A combination of (3.18) and (3.19), we get

$$\|w_{T^r}\|_r^r \leq \frac{1}{1 - \mu} (\|v\|_r^r - \alpha \|v\|_1^1 - (\|u\|_r^r - \alpha \|u\|_1^1) + 2\|w_{T^r}\|_r^r) + \frac{\rho}{1 - \mu} \|\Phi w\|_2. \quad (3.20)$$

By utilizing (3.18) and combining with (3.20), we get

$$\|w\|_r^r + \alpha \|w\|_1^1 \leq \|w_{T^r}\|_r^r + \alpha \|w\|_1^1 + \|w_{T^r}\|_r^r \leq (1 + \mu)\|w_{T^r}\|_r^r + \rho \|\Phi w\|_2$$

$$\leq \frac{1 + \mu}{1 - \mu} (\|v\|_r^r - \alpha \|v\|_1^1 - (\|u\|_r^r - \alpha \|u\|_1^1) + 2\|w_{T^r}\|_r^r) + \frac{2\rho}{1 - \mu} \|\Phi w\|_2. \quad (3.17)$$

The proof is finished. \(\square\)

Now we give the characterization of robust reconstruction for the signal employing the method (1.2) based on the weighted $\ell_r - \ell_1$ robust NSP.

**Theorem 3.3.** Let $x$ be the real-world signal in $b = \Phi x + \epsilon$ with $\|\epsilon\|_2 \leq \eta$, $\hat{x}$ be the solution to $b = \Phi x + \epsilon$ with $\|\epsilon\|_2 \leq \xi$. Suppose that the matrix $\Phi$ fulfills the weighted $\ell_r - \ell_1$ robust NSP of order $k$ with constants $\mu \in (0, 1)$ and $\rho \in (0, +\infty)$. We have

$$\|x - \hat{x}\|_r^r \leq \frac{2(1 + \mu)}{1 - \mu} \sigma_k(x)^r + \frac{2\rho}{1 - \mu}(\eta + \xi). \quad (3.21)$$

**Remark 3.4.** The conclusion shows that the upper bound of the reconstruction error is controlled by the optimal $k$-term approximation error and the noise level.

**Proof.** Observing that $\|\Phi(x - \hat{x})\|_2 \leq \eta + \xi$, the remainder of proof is parallel to that of Theorem 2.6. \(\square\)

In what follows, we concentrate on another weighted $\ell_r - \ell_1$ robust NSP, that is, the $\ell_q$ weighted $\ell_r - \ell_1$ robust NSP for $1 \leq q \leq 2$.

**Definition 3.5.** For any set $T \subseteq [n]$ with $|T| \leq k$, the matrix $\Phi$ is said to fulfill the $\ell_q$ weighted $\ell_r - \ell_1$ robust null space property of order $k$ with constants $\mu \in (0, 1)$ and $\rho \in (0, +\infty)$ if

$$\|w_T\|_q + \alpha \|w\|_1 \leq \frac{\mu}{k^r} \|w_{T^r}\|_r^r + \rho \|\Phi w\|_2 \quad (3.22)$$

for all $w \in \mathbb{R}^n$. 
Now we provide the description of the robust recovery of the signal with (1.2) under the $\ell_q$ weighted $\ell_r - \ell_1$ robust NSP.

**Theorem 3.6.** Let $x$ be the real-world signal in $b = \Phi x + \epsilon$ with $\|\epsilon\|_2 \leq \eta$, $\tilde{x}$ be the solution to $b = \Phi x + \epsilon$ with $\|\epsilon\|_2 \leq \xi$. Suppose that the matrix $\Phi$ fulfills the $\ell_q$ weighted $\ell_r - \ell_1$ robust NSP of order $k$ with constants $0 < \mu < \left(\frac{(1 - \alpha)/(1 + \alpha)}{1}ight)^{1/r}$ and $\rho \in (0, +\infty)$. We have

$$\|x - \tilde{x}\|_q \leq \frac{C}{k^{\frac{2}{q} - \frac{2}{r}}} \sigma_k(x)_r + D(\eta + \xi),$$

where

$$C = \frac{2^{\frac{2}{q}} - 1(1 + \mu)(1 + \mu^r)^{\frac{1}{q}}}{(1 - \alpha - (1 + \mu)(1 + \mu^r)^{\frac{1}{q}})}, \quad D = \frac{2^{\frac{2}{q}} - 1(1 + \mu)}{(1 - \alpha - (1 + \mu)(1 + \mu^r)^{\frac{1}{q}}) + \rho}.$$

**Proof.** Set $w = x - \tilde{x}$. We decompose $w$ into $w = w_T + w_T^\perp$ with $|T| \leq k$. Rearrange the indices such that $|w_T[1]| \geq |w_T[2]| \geq \cdots \geq |w_T[k]| \geq |w_T[1]| \geq |w_T[2]| \geq \cdots$. Then,

$$\|w_T\|_q \leq \sum_{i \geq 1} |w_T[i]|^{q-r} |w_T[i]|^r \leq |w_T[k]|^{q-r} \sum_{i \geq 1} |w_T[i]|^r \leq \left(\frac{1}{k}\|w_T\|_r\right)^{\frac{q-r}{r}} \|w_T\|_r \leq \frac{1}{k^{\frac{2}{q} - \frac{2}{r}}} \|w\|_q.$$

That is,

$$\|w_T\|_q \leq \frac{1}{k^{\frac{2}{q} - \frac{2}{r}}} \|w\|_r.$$

Observing that

$$\|w_T\|_r \leq k^{\frac{1}{q} - \frac{1}{2}} \|w_T\|_q,$$

combining with (3.22), we get

$$\|w_T\|_r \leq \mu \|w_T\|_r + \rho k^{\frac{1}{q} - \frac{1}{2}} \|\Phi w\|_2,$$

which implies

$$\|w_T\|_r^r \leq \mu^r \|w_T\|_r^r + \rho^r k^{1 - \frac{2}{q}} \|\Phi w\|_2^r.$$

By Lemma 3.7 and (3.26), it leads to

$$\|w_T\|_r^r \leq \|w_T\|_r^r + \alpha \|w\|_1^r + 2 \|x_T\|_r^r \leq \mu^r \|w_T\|_r^r + \rho^r k^{1 - \frac{2}{q}} \|\Phi w\|_2^r + \alpha \|w\|_1^r + 2 \|x_T\|_r^r,$$

which deduces that

$$\|w_T\|_r^r \leq \frac{1}{1 - \mu^r} \left(\rho^r k^{1 - \frac{2}{q}} \|\Phi w\|_2^r + \alpha \|w\|_1^r + 2 \|x_T\|_r^r\right).$$
Combining with (3.26) and (3.27), we get
\[
\|w\|_r = \|w_T\|_r^\alpha + \|w_T\|_r^\beta \leq (1 + \mu^r)\|w_T\|_r^\alpha + \rho^r k^1 - \frac{\gamma}{4}\|\Phi w\|_2^\alpha
\]
\[
\leq \frac{2(1 + \mu^r)}{1 - \mu^r}\|x_T\|_r^\alpha + \frac{2\rho^r k^1 - \frac{\gamma}{4}}{1 - \mu^r}\|\Phi w\|_2^\alpha + \frac{(1 + \mu^r)\alpha}{1 - \mu^r}\|w\|_r^\alpha.
\]
Noting that \(\mu < ((1 - \alpha)/(1 + \alpha))^{1/r}\) and \(\|w_T\|_r^\alpha \leq \|w_T\|_r^\beta\), then
\[
\|w\|_r \leq \frac{2(1 + \mu^r)}{1 - \alpha - (1 + \alpha)\mu^r}\|x_T\|_r^\alpha + \frac{2\rho^r k^1 - \frac{\gamma}{4}}{1 - \alpha - (1 + \alpha)\mu^r}\|\Phi w\|_2^\alpha,
\]
which implies
\[
\|w\|_r \leq \frac{2^{\frac{1}{r}} - 1}{k^{\frac{2}{r}} - \frac{\gamma}{4}}\left(\frac{2^{\frac{1}{r}}(1 + \mu^r)^{\frac{1}{r}}}{1 - \alpha - (1 + \alpha)\mu^r}\|x_T\|_r + \frac{2^{\frac{1}{r}}(1 + \mu^r)^{\frac{1}{r}}}{1 - \alpha - (1 + \alpha)\mu^r}\|\Phi w\|_2\right). \tag{3.28}
\]
For \(1 \leq q \leq 2\), by (3.22) and (3.25), we obtain
\[
\|w\|_q \leq \|w_T\|_q^\alpha + \|w_T\|_q^\beta
\]
\[
\leq \frac{1}{k^{\frac{2}{r}} - \frac{\gamma}{4}}\|w\|_r^\alpha + \frac{\mu}{k^{\frac{2}{r}} - \frac{\gamma}{4}}\|w_T\|_r + \rho\|\Phi w\|_2
\]
\[
\leq \frac{1 + \mu}{k^{\frac{2}{r}} - \frac{\gamma}{4}}\|w\|_r + \rho\|\Phi w\|_2. \tag{3.29}
\]
By the feasibility of \(\hat{x}\), we get
\[
\|\Phi w\|_2 = \|\Phi x - \Phi \hat{x}\|_2 \leq \|\Phi x - b\|_2 + \|\Phi \hat{x} - b\|_2 \leq \eta + \xi. \tag{3.30}
\]
Plugging the equalities (3.28) and (3.30) into (3.29), the desired result is derived. \(\Box\)

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**Appendix A. Appendices**

**Lemma 3.7.** For all vectors \(u, v \in \mathbb{R}^n\) and \(T \subseteq [n]\), we have
\[
\|(u - v)_T\|_r^\alpha \leq \|v\|_r^\alpha - \alpha\|v\|_1^\beta - (\|u\|_r^\alpha - \alpha\|u\|_1^\beta) + \|(u - v)_T\|_r^\alpha + \alpha\|u - v\|_1^\beta + 2\|w_T\|_r^\alpha. \tag{3.31}
\]
\textbf{Proof.} By applying the $r$-triangular inequality, we get
\begin{equation}
\| (u-v)_{T^c} \|^r_r \leq \| u_{T^c} \|^r_r + \| v_{T^c} \|^r_r \label{3.32}
\end{equation}
and
\begin{equation}
\| u \|^r_r = \| u_{T^c} \|^r_r + \| u_T \|^r_r \leq \| u_{T^c} \|^r_r + \| (u-v)_T \|^r_r + \| v_T \|^r_r. \label{3.33}
\end{equation}
By using the inequality $(a+b)^r \leq a^r + b^r$ for $a, b \geq 0$ and $0 < r \leq 1$, we get
\begin{equation}
\|v\|_1^r - \|u\|_1^r \leq \|u - v\|_1^r + \|u\|_1^r - \|u\|_1^r \leq \|u - v\|_1^r. \label{3.34}
\end{equation}
Combining with (3.32)-(3.34), we obtain the desired result. \hfill \Box

\textbf{Lemma 3.8.} \cite{22} For all vectors $u, v \in \mathbb{R}^n$ supported on disjoint sets $T, S \subseteq [n]$ with $|T| \leq k, |S| \leq s$,
\begin{equation}
\langle \Phi u, \Phi v \rangle \leq \delta_{k+s} \|u\|_2 \|u\|_2 \label{3.35}
\end{equation}
holds.

\textbf{Proof of Theorem 2.8.} For any $w \in N(\Phi)$, we think about the decomposition of $w$: $w = w_T + w_{T^c}$ with $|T| \leq k$. Set $T^c = T_1 \cup T_2 \cup \cdots$, where $T_1$ is the index set of $k$ largest absolute elements of $w$ in $T^c$, $T_2$ is the index set of $k$ largest absolute elements of $w$ in $(T \cup T_1)^c$ and so forth. Set $T_0 = T \cup T_1$. It is easy to check that $|\text{supp}(w_{T_i})| + |\text{supp}(w_{T_j})| \leq 2k$ and $\langle w_{T_i}, w_{T_j} \rangle = 0$ for $i \neq j$. By applying Lemma 3.8, we gain $\langle \Phi w_{T_i}, \Phi w_{T_j} \rangle \leq \delta_{2k} \|w_{T_i}\|_2 \|w_{T_j}\|_2$. Observing that $\Phi w = 0$, we get $\Phi (w_T + w_{T_1}) = - \sum_{j \geq 2} \Phi w_{T_j}$, and
\begin{equation}
\|\Phi (w_T + w_{T_1})\|_2^2 = \left| \left\langle \Phi (w_T + w_{T_1}), \sum_{j \geq 2} \Phi w_{T_j} \right\rangle \right| \\
\leq \sum_{j \geq 2} \delta_{2k} \|w_{T_1}\|_2 \|w_{T_j}\|_2 + \sum_{j \geq 2} \delta_{2k} \|w_{T_1}\|_2 \|w_{T_j}\|_2 \\
= \delta_{2k} (\|w_T\|_2 + \|w_{T_1}\|_2) \sum_{j \geq 2} \|w_{T_j}\|_2. \label{3.36}
\end{equation}
By using the RIP of order $k$ of $\Phi$ and (3.36), it leads to
\begin{equation}
\|w_T\|_2^2 + \|w_{T_1}\|_2^2 = \|w_T + w_{T_1}\|_2^2 \leq \frac{1}{1 - \delta_{2k}} \|\Phi w_{T_1} \|_2^2 \\
\leq \frac{\delta_{2k}}{1 - \delta_{2k}} (\|w_T\|_2 + \|w_{T_1}\|_2) \sum_{j \geq 2} \|w_{T_j}\|_2.
\end{equation}
Setting $\psi = \frac{\delta_{2k}}{1 - \delta_{2k}} \sum_{j \geq 2} \|w_{T_j}\|_2$, it implies
\begin{equation}
(\|w_T\|_2 - \frac{1}{2} \psi)^2 + (\|w_{T_1}\|_2 - \frac{1}{2} \psi)^2 \leq \frac{1}{2} \psi^2,
\end{equation}
which results in
\begin{equation}
\|w_T\|_2 \leq \frac{1 + \sqrt{2}}{2} \psi = \frac{(1 + \sqrt{2}) \delta_{2k}}{2(1 - \delta_{2k})} \sum_{j \geq 2} \|w_{T_j}\|_2. \label{3.37}
\end{equation}
By the definition of $T_j$, we get
\[ \|w_T\|_2 \leq \frac{1}{k^{\frac{1}{2} - \frac{1}{2}}} \|w_{T_{j-1}}\|_r \]
for $j \geq 2$, it brings
\[
\sum_{j \geq 2} \|w_{T_j}\|_2 \leq \left(\sum_{j \geq 2} \|w_{T_j}\|_2^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{k^{1 - \frac{1}{2}}} \sum_{j \geq 1} \|w_{T_j}\|_r^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{k^{1 - \frac{1}{2}}} \|w_{T}\|_r^r\right)^{\frac{1}{2}} = \frac{1}{k^{\frac{1}{2} - \frac{1}{2}}} \|w_{T}\|_r. \tag{3.38}
\]
Plugging (3.38) into (3.37), combining with Hölder inequality, we get
\[
\|w_T\|_r^r \leq k^{1 - \frac{1}{r}} \|w_T\|_2^r \leq \frac{(1 + \sqrt{2})^r \delta^r_{2k}}{2^r \left(1 - \frac{1}{r} \right)^r} \|w_{T_c}\|_r^r. \tag{3.39}
\]
One can easily verify that
\[
\alpha' \|w\|_1^1 \leq \alpha' \|w_{T_c}\|_1^1 + \alpha' \|w_T\|_1^r \tag{3.40}
\]
with $0 < \alpha' < 1$. Combining with (3.39) and (3.40), we derive
\[
\|w_T\|_r^r + \frac{\alpha'}{1 - \alpha'} \|w\|_1^r \leq \frac{1}{1 - \alpha'} \left(\frac{(1 + \sqrt{2})^r \delta^r_{2k}}{2^r \left(1 - \frac{1}{r} \right)^r} + \alpha'\right) \|w_{T_c}\|_r^r. \tag{3.41}
\]
Set $\frac{\alpha'}{1 - \alpha'} = \alpha$, under the condition of (2.14), the above equation (3.41) deduces that $\Phi$ meets the weighted $\ell_r - \ell_1$ stable NSP of order $k$ with constant $\mu$, where
\[
\mu = \frac{(1 + \alpha)(1 + \sqrt{2})^r \delta^r_{2k}}{2^r \left(1 - \frac{1}{r} \right)^r} + \alpha.
\]

References


