A new sufficient condition for non-convex sparse recovery via weighted $\ell_r - \ell_1$ minimization

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Abstract. In this letter, we discuss the reconstruction of sparse signals from undersampled data, which belongs to the core content of compressed sensing. A new sufficient condition in terms of the restricted isometry constant (RIC) and restricted orthogonality constant (ROC) is first established for the performance guarantee of recently proposed non-convex weighted $\ell_r - \ell_1$ minimization in recovering (approximately) sparse signals that may be polluted by noise. To be specific, it is shown that if the RIC $\delta_s$ and ROC $\theta_{s,s}$ of measurement matrix obey

$$\delta_s + \nu(s)\theta_{s,s} < 1,$$

where $\nu(s)$ depends on $s$ for given quantities, then any $s$-sparse signals in noiseless setting are guaranteed to be recovered accurately via solving the constrained weighted $\ell_r - \ell_1$ minimization optimization problem and any (approximately) $s$-sparse signals can be estimated robustly in the noisy case. In addition, we provide several pivotal remarks which indicate the recovery guarantee is much less restricted than the existing one. The results obtained contribute to proving the fidelity of the excellent weighted $\ell_r - \ell_1$ minimization method.

Key words. Compressed sensing; Constrained weighted $\ell_r - \ell_1$ minimization; Restricted isometry property; Nonconvex sparse recovery

1 Introduction

Recovering a high-dimensional sparse signal in the awareness of significantly fewer observations, probably perturbed by noise, is an essential topic in signal processing. This as well as other associating topics in compressed sensing [1–3] have received plenty of recent attention in a diverse set of areas, incorporating photography, holography, facial recognition, magnetic resonance imaging, etc.

In compressed sensing, one discusses the model below

$$y = Ax + e,$$  \hspace{1cm} (1.1)
where \( y \in \mathbb{R}^m \), \( A \in \mathbb{R}^{m \times n} \) (\( m \ll n \)) is a given measurement matrix, \( x \in \mathbb{R}^n \) is an original sparse signal, and \( e \in \mathbb{R}^m \) is an additive noise. The objective is to estimate the sparse signal \( x \) in the knowledge of the measurement matrix \( A \) and the observation vector \( y \). The signal \( x \) could be recovered by the well-known \( \ell_1 \) minimization method suggested through Candès and Tao \([1]\)

\[
\min_{x \in \mathbb{R}^n} \|z\|_1 \text{ subject to } \|Ax - y\|_2 \leq \epsilon,
\]

where \( \epsilon > 0 \) denotes the noise level. Especially, one takes \( \epsilon = 0 \) in the noise-free situation. One favourably employs this approach as an efficiency method for recovering a sparse signal in a wide range of scenarios, see, e.g. \([5, 10, 20]\).

Recently, Zhou and Yu \([4]\) introduced to substitute (1.2) for the recovery of sparse signal \( x \) is to think over the constrained weighted \( \ell_r - \ell_1 \) minimization model below

\[
\hat{x} = \arg \min_{z \in \mathbb{R}^n} \left\{ \|z\|_r^r - \alpha \|z\|_1 \text{ subject to } \|Az - y\|_2 \leq \epsilon \right\},
\]

where \( \|z\|_r^r = \sum_{i=1}^n |z_i|^r \), \( 0 < r \leq 1 \), \( 0 \leq \alpha \leq 1 \). Suppose \( \alpha \neq 1 \) in the case of \( r = 1 \) henceforth. It is easy to see that (1.3) goes to the conventional \( \ell_r \) minimization model as \( \alpha = 0 \). They demonstrated that the (approximately) sparse signals could be accurately/robustly reconstructed with the model (1.3) in the noiseless/noisy situation under the \( r \)-restricted isometry property condition. Furthermore, with a number of numerical simulations, they indicated that the weighted \( \ell_r - \ell_1 \) minimization performs better than the other present classic methods, such as ADMM-Lasso \([21]\), CoSaMP \([22]\), IHT \([23]\) and \( \ell_1 - 2 \) minimization \([24]\). In \([4]\), the researchers analyzed in detail some other virtues from the constrained weighted \( \ell_r - \ell_1 \) minimization.

One of the highly extensively utilized criterion for the recovery of sparse signal is the restricted isometry property (RIP) proposed by Candès and Tao \([1]\). A vector \( u \) is \( s \)-sparse in the case that \( \#\text{supp}(u) \leq s \), in which \( \text{supp}(u) = \{ j : u_j \neq 0 \} \) is the support of \( u = (u_1, u_2, \ldots, u_n)^\top \in \mathbb{R}^n \).

**Definition 1.1.** For an \( m \times n \) matrix \( A \) and an integer \( s \) with \( 1 \leq s \leq n \), the restricted isometry constant (RIC) of order \( s \) is the smallest constant such that for every \( s \)-sparse vector \( u \), the following inequalities hold

\[
(1 - \delta_s)\|u\|_2^2 \leq \|Au\|_2^2 \leq (1 + \delta_s)\|u\|_2^2.
\]

Let \( t \) be an integer with \( 1 \leq t \leq n \). If \( s + t \leq n \), the restricted orthogonality constant (ROC) of order \((s, t)\) is the smallest number that obeys

\[
|\langle Au, Av \rangle| \leq \theta_{s,t} \|u\|_2 \|v\|_2
\]

for every \( s \)-sparse vector \( u \) and \( t \)-sparse vector \( v \) such that the supports of \( u \) and \( v \) are disjoint.

In the references, researchers have introduced a lot of sufficient conditions regarding the RIP conditions for accurate/robust reconstruction of sparse signals exploiting the models (1.2) and (1.3), see, e.g. \([5–17]\). Especially, Tony Cai and Anru Zhang \([7]\) showed that the condition \( \delta_s + \theta_{s,s} < 1 \) can ensure the accurate construction of every \( s \)-sparse signal in the noise-free situation using the \( \ell_1 \) minimization. Simultaneously, for any \( \epsilon > 0 \),
Lemma 2.1. Let $s, t \leq n$ and $\rho \geq 0$. Assume that the supports of $u, v \in \mathbb{R}^n$ are disjoint and $\# \text{supp}(u) \leq s$. If $\|v\|_1 \leq pt$ and $\|v\|_\infty \leq \rho$, then
\[
\langle Au, Av \rangle \leq \theta_{s,t} \|u\|_2 \cdot \rho \sqrt{t}.
\]  
(2.6)

We then need the result below given by [7]. It presents an inequality respecting the sums of two arrays of the $q$th power of numbers greater than or equal to 0 in the knowledge of the inequality of their sums.

Lemma 2.2. Assume that $\mu \geq 0$, $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, and $\sum_{i=1}^s a_i + \mu \geq \sum_{i=s+1}^n a_i$, then for every $q \geq 1$,
\[
\sum_{i=s+1}^n a_i^q \leq s \left( \sqrt[ q]{ \frac{ \sum_{i=1}^s a_i^q }{s} + \frac{ \mu }{s} } \right)^q.
\]  
(2.7)
Lemma 2.3. Let \( \hat{x} \) be the solution of (1.3), recovery error \( h = \hat{x} - x \), \( T = \text{supp}(x_{\max(s)}) \) and \( S = \text{supp}(h_{\max(s)}) \). Then,

\[
\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + 2\|x_T\|_r^r + \alpha\|h\|_1^r. \tag{2.8}
\]

Proof. Since \( \hat{x} \) is a minimizer of (1.3), we get

\[
\|x + h\|_r^r - \alpha\|x + h\|_1^r \leq \|x\|_r^r - \alpha\|x\|_1^r.
\]

That is,

\[
\|x_T + h_T\|_r^r + \|x_T^c + h_{T^c}\|_r^r \leq \|x_T\|_r^r + \|x_T^c\|_r^r + \alpha\|x + h\|_1^r - \alpha\|x\|_1^r. \tag{2.9}
\]

Observing that \( \|h_T\|_r \leq \|h_S\|_r \) and \( \|h_{S^c}\|_r \leq \|h_{T^c}\|_r \), combining with \( r \)-inverse triangular inequality, the left side (LS) of (2.9) yields

\[
\begin{align*}
LS & \geq \|x_T\|_r^r - \|h_T\|_r^r + \|h_{T^c}\|_r^r - \|x_T^c\|_r^r \\
& \geq \|x_T\|_r^r - \|h_S\|_r^r + \|h_{S^c}\|_r^r - \|x_T^c\|_r^r. \tag{2.10}
\end{align*}
\]

As for the right side (RS) of (2.9), by using the fact that \((a + b)^r \leq a^r + b^r \) for \( a, b \geq 0 \), one gets

\[
RS \leq \|x_T\|_r^r + \|x_T^c\|_r^r + \alpha\|h\|_1^r. \tag{2.11}
\]

Hence, together with (2.9)-(2.11), it implies

\[
\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + 2\|x_T\|_r^r + \alpha\|h\|_1^r.
\]

\[\square\]

3 Main results

On the basis of the aforementioned preparations, we now give the main results in this section.

Theorem 3.1. Set \( \nu(s) = \left( \frac{2^{s-2}a_{s}(1+2^{s-2})}{s^{s-2}a_{s}^{2}+2^{s-2}} + \frac{1}{2^{s-1}} \right) \). Think out the signal reconstruction model (1.1) with \( \|e\|_2 \leq \eta \) for some \( \eta \geq \epsilon \). Assume that \( \hat{x} \) is the solution of (1.3) with \( \|e\|_2 \leq \epsilon \). If

\[
\delta_s + \nu(s)\theta_{s,s} < 1, \tag{3.12}
\]

then

\[
\|\hat{x} - x\|_2 \leq C_1\|x_T^c\|_r + C_2(\epsilon + \eta), \tag{3.13}
\]

where

\[
C_1 = \lambda^{-1}(s) \left\{ \frac{2^{s-2}\theta_{s,s}(1+2^{s-2})}{s^{s-2}a_{s}^{2}(1-\delta_s-2^{s-1}\theta_{s,s})} + \frac{2^{s-3}}{s^{s-2}} \right\}.
\]
and

\[ C_2 = \frac{(1 + 2^{\frac{s}{2}} - 2)^{\frac{1}{2}} \sqrt{1 + \delta_s}}{\lambda(s)(1 - \delta_s - 2^{\frac{s}{2}} - 1 \theta_{s,s})} \]

with

\[ \lambda(s) = \left\{ 1 - \sqrt{\frac{2^{\frac{s}{2}} - 2 \alpha \theta_{s,s}}{s^{\frac{1}{2}} - \frac{1}{2} (1 - \delta_s - 2^{\frac{s}{2}} - 1 \theta_{s,s})} + \frac{2^{\frac{s}{2}} - 3 \alpha \theta_{s,s}}{s^{\frac{1}{2}} - \frac{1}{2}}} \right\}. \]

**Remark 3.2.** When \( \alpha = 0 \) and \( r = 1 \), Theorem 3.1 reduces to Theorem 2.2 [7]. Since the condition (3.12) blends RIC \( \delta_s \) and ROC \( \theta_{s,s} \), it appears slightly complex. Surprisingly, by applying Lemma 3.1 [7], we can derive a brief condition as to (1.3), which is as follows

\[ \delta_s \leq \begin{cases} \frac{1}{1 + 2^{\nu(s)}} & \text{if } s \text{ is even, } s \geq 2; \\ \sqrt{\frac{s-1}{s^{\nu(s)} - 1 + 2^{\nu(s)}}} & \text{if } s \text{ is odd, } s \geq 3. \end{cases} \] \hspace{1cm} (3.14)

It should be emphasized that if we display (3.12) by (3.14), then Theorem 3.1 as before holds.

**Remark 3.3.** Similar to what was done in Remark 3.2, by using the mutual coherence, we can also gain a concise condition for (1.3). The definition of mutual coherence (see, e.g. [14, 18]) is

\[ \mu = \max_{i \neq j} |A_i^T A_j|, \]

where \( A_i(i = 1, \cdots, n) \) is the \( i \)th column of \( A \), and \( \|A_i\|_2 = 1, \ i = 1, \cdots, n. \) In addition, it is known from [18] that \( \delta_s \leq (s - 1)\mu \) and \( \theta_{s,t} \leq \sqrt{s} \mu. \) From (3.12), we can obtain

\[ \mu < \frac{1}{s - 1 + s \nu(s)}. \] \hspace{1cm} (3.15)

Theorem 3.1 also holds under the condition (3.15). Newly, the reference [19] also presented a sufficient condition based on mutual coherence with (1.3), whose expression is

\[ \mu < \frac{1}{s - 1 + 2^{\nu(s)}}. \] \hspace{1cm} (3.16)

To compare the condition we established with that of [19], their upper bounds are given in Fig. 3.1. In the experiment, we take \( \alpha = 0.2 \) and \( n = 100. \) Observing Fig. 3.1, we can see that our conditions are better than theirs except for a few ranges of values of \( s \) and \( r. \)

**Remark 3.4.** Theorem 3.1 still holds (only replace \( n \) in \( \nu(s) \) and \( \lambda(s) \) by 1) provided that we substitute the model (1.3) with the following model

\[ \hat{x} = \arg \min \{\|z\|_r - \alpha \|z\|_2 \text{ subject to } \|Az - y\|_2 \leq \epsilon\}. \] \hspace{1cm} (3.17)

Theorem 3.1 now returns to Theorem 1 [12] in the case of \( \alpha = 1 \) and \( r = 1. \) The resulting condition for (3.17) is

\[ \delta_s + \frac{\sqrt{s} + \sqrt{2} - 1}{\sqrt{s} - 1} \theta_{s,s} < 1. \] \hspace{1cm} (3.18)

A comparison of condition (3.18) with that of [14] has been done in [12] and they showed that condition (3.18) is weaker than that of [12].
4 Proof

Proof of Theorem 3.1. Let $h = \hat{x} - x$. By Lemma 2.3, we get

$$
\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + 2\|x_{T^c}\|_r^r + \alpha \|h\|_1^r.
$$

In accordance with the feasibility of $\hat{x}$,

$$
\|Ah\|_2 \leq \|Ax - y\|_2 + \|A\hat{x} - y\|_2 \leq \epsilon + \eta.
$$

By employing Hölder inequality, we get

$$
\|h_{S^c}\|_\infty \leq \frac{\|h_S\|_r^r}{s} \leq \left(\frac{s^\frac{1}{r} - \frac{1}{2}}{s^\frac{1}{r}}\right) \|h_S\|_r^r \leq \frac{\|h_S\|_s^s}{s^\frac{1}{r}} = \frac{\|h_S\|_r^r}{s^\frac{1}{r}},
$$

which yields

$$
\|h_{S^c}\|_\infty \leq \frac{\|h_S\|_s^s}{s^\frac{1}{r}} \leq \frac{2^\frac{1-r}{2}\|h_S\|_r^r}{s^\frac{1}{r}} + \frac{2^{\frac{1-r}{2}-2}(2^\frac{1}{r}\|x_{T^c}\|_r^r + \alpha \|h\|_1^r)}{s^\frac{1}{r}}.
$$

Applying Lemma 2.2 to (4.19) with $q = 1/r$ and $\mu = 2\|x_{T^c}\|_r^r + \alpha \|h\|_1^r$ deduces

$$
\|h_{S^c}\|_1 \leq s \left(\frac{\|h_S\|_s^s}{s} + \frac{2\|x_{T^c}\|_r^r + \alpha \|h\|_1^r}{s^\frac{1}{r}}\right)^{\frac{1}{r}}.
$$

(a) $\leq s2^{\frac{1}{r}-1} \left(\frac{\|h_S\|_s^s}{s} + \frac{2\|x_{T^c}\|_r^r + \alpha \|h\|_1^r}{s^\frac{1}{r}}\right)^{\frac{1}{r}}$

(b) $\leq s2^{\frac{1}{r}-1} \left(\frac{\|h_S\|_s^s}{s^\frac{1}{r}} + \frac{2\|x_{T^c}\|_r^r + \alpha \|h\|_1^r}{s^\frac{1}{r}}\right)^{\frac{1}{r}}$

$$
\leq s \left(\frac{2^{\frac{1-r}{2}}\|h_S\|_r^r}{s^\frac{1}{r}} + \frac{2^{\frac{1-r}{2}-2}(2^\frac{1}{r}\|x_{T^c}\|_r^r + \alpha \|h\|_1^r)}{s^\frac{1}{r}}\right),
$$

where (a) is from the fact $(u_1 + u_2)^{1/r} \leq 2^{\frac{1}{r}-1}(u_1 + u_2)$ for any $u_1, u_2 \geq 0$, and (b) is due to Cauchy-Schwarz inequality. It then follows from Lemma 2.1 that

$$
\langle Ah_S, Ah_{S^c}\rangle \leq \|h_S\|_2 \left(\frac{2^{\frac{1-r}{2}}\|h_S\|_r^r}{s^\frac{1}{r}} + \frac{2^{\frac{1-r}{2}-2}(2^\frac{1}{r}\|x_{T^c}\|_r^r + \alpha \|h\|_1^r)}{s^\frac{1}{r}}\right).
$$

Fig. 3.1: Comparison conditions (3.15) and (3.16) for (a) $r = 3/4$, (b) $s = 5$. 

(a) (b)
As a result, we have
\[ |\langle Ah, Ah_S \rangle| = |\langle Ah_S, Ah_S \rangle + \langle Ah_S, Ah_{S^c} \rangle| \]
\[ \geq (1 - \delta_s)\|h_S\|_2^2 - |\langle Ah_S, Ah_{S^c} \rangle| \]
\[ \geq (1 - \delta_s)\|h_S\|_2^2 - \theta_s,\|h_S\|_2 \left( 2^{\frac{3}{2} - 1}\|h_S\|_2^2 + \frac{2^{\frac{3}{2} - 2}\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2} - \frac{1}{2}}} \right). \] \tag{4.24}

On the other hand, by (4.20) and the definition of RIC, we get
\[ |\langle Ah_S, Ah \rangle| \leq \|Ah\|_2\|Ah_S\|_2 \leq (\epsilon + \eta)\sqrt{1 + \delta_s}\|h_S\|_2. \] \tag{4.25}

Combining with (4.24) and (4.25), it leads to
\[ (1 - \delta_s - 2^{\frac{1}{2} - 1}\theta_s,\|h_S\|_2 - \theta_s,\|h_S\|_2 \left( 2^{\frac{3}{2} - 2}\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1 \right) \leq (\epsilon + \eta)\sqrt{1 + \delta_s}. \] \tag{4.26}

Under the condition (3.12), it results in \( 1 - \delta_s - 2^{\frac{1}{2} - 1}\theta_s,\|h_S\| > 0 \), and thereby yields
\[ \|h_S\|_2 \leq \frac{1}{1 - \delta_s - 2^{\frac{1}{2} - 1}\theta_s,\|h_S\| \left( 2^{\frac{3}{2} - 2}\theta_s,\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1 \right)s^\frac{1}{2} - \frac{1}{2}\left( \frac{2^{\frac{3}{2} - 2}\theta_s,\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2}} - \frac{1}{2}} \right) + \left( \epsilon + \eta \right)\sqrt{1 + \delta_s}. \] \tag{4.27}

Exploiting again Lemma 2.2 to (4.19) with \( q = 2/r \), it brings
\[ \|h_{S^c}\|_2 \leq \|h_S\|_2 + \frac{2^{\frac{3}{2} - 2}\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2}} - \frac{1}{2}}. \] \tag{4.28}

Combining with (4.27) and (4.28), we have
\[ \|h\|_2 = \sqrt{\|h_S\|_2^2 + \|h_{S^c}\|_2^2} \]
\[ \leq \left\{ \|h_S\|_2^2 + \frac{2^{\frac{3}{2} - 1}\|h_S\|_2^2 + \frac{2^{\frac{3}{2} - 2}\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2}} - \frac{1}{2}} \right\}^\frac{1}{2} \]
\[ \leq (1 + 2^{\frac{3}{2} - 2})^\frac{1}{2}\|h_S\|_2 + \frac{2^{\frac{3}{2} - 3}\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2}} - \frac{1}{2}} \]
\[ \leq \frac{(1 + 2^{\frac{3}{2} - 2})^\frac{1}{2}}{1 - \delta_s - 2^{\frac{1}{2} - 1}\theta_s,\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1} + \left( \epsilon + \eta \right)\sqrt{1 + \delta_s} \]
\[ + \frac{2^{\frac{3}{2} - 3}\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2}} - \frac{1}{2}}, \]

which implies
\[ \lambda(s)\|h\|_2 = \left\{ 1 - \sqrt{\frac{2^{\frac{3}{2} - 2}\alpha \frac{1}{2}\theta_s,\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1}{s^{\frac{1}{2}} - \frac{1}{2}}} + \frac{2^{\frac{3}{2} - 3}\alpha \frac{1}{2}}{s^{\frac{1}{2}} - \frac{1}{2}} \right\} \|h\|_2 \]
\[ \leq \left\{ \frac{2^{\frac{3}{2} - 2}\theta_s,\|x_{T^c}\|_r + \frac{2^{\frac{3}{2} - 3}}{s^{\frac{1}{2}} - \frac{1}{2}} \right\} \|x_{T^c}\|_r \]
\[ + \frac{(1 + 2^{\frac{3}{2} - 2})^\frac{1}{2}}{1 - \delta_s - 2^{\frac{1}{2} - 1}\theta_s,\|x_{T^c}\|_r + \alpha \frac{1}{2}\|h_1\|_1} (\epsilon + \eta). \] \tag{4.29}
Making use of the condition (3.12), we obtain the upper bound estimation of recovery error

$$
\| h \|_2 \leq \lambda^{-1}(s) \left\{ \frac{2^{3/2} - 2\theta_{s,s}(1 + 2^{1/2} - 2^{1/2})^{1/2}}{s^{1/2} - 1/4 (1 - \delta_{s} - 2^{1/2 - 1}\theta_{s,s})} + \frac{2^{3/2 - 3}}{s^{1/2 - 1/4}} \right\} \| x_T^- \|_r \\
+ \frac{(1 + 2^{1/2 - 2})^{1/2}}{\lambda(s)(1 - \delta_{s} - 2^{1/2 - 1}\theta_{s,s})} (\epsilon + \eta).
$$

(4.30)

\[ \square \]

5 Conclusion

In this paper, we establish the proof for the performance guarantee of weighted $\ell_r - \ell_1$ minimization in recovering sparse signals which are possibly disturbed by noise. This work partially fills up the void of combining RIP and ROP, two powerful theoretical tools, to construct sufficient conditions for weighted $\ell_r - \ell_1$ minimization method to accurately/robustly recover sparse signals. Note that our current work only obtains a loose recovery condition and a recovery error bound. One future direction is to provide tighter results even the sharp ones.

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